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Probability Distribution of Curvatures of Isosurfaces in Gaussian Random Fields

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Abstract

An expression for the joint probability distribution of the principal curvatures at an arbitrary point in the ensemble of isosurfaces defined on isotropic Gaussian random fields on $\mathbb{R}^n$ is derived. The result is obtained by deriving symmetry properties of the ensemble of second derivative matrices of isotropic Gaussian random fields akin to those of the Gaussian orthogonal ensemble.

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I. INTRODUCTION

We closely follow the notation in [1] and [2]. Let $\mathbf{T}$ be a tensor with rank a and dimensions $(d_1, \ldots, d_a)$. The bijective linear map vec associates $\mathbf{T}$ to the vector $\operatorname{vec}\mathbf{T}$ in $\mathbb{R}^N$, $N = \prod_{i=1}^a d_i$, with entry $(\operatorname{vec}\mathbf{T})_k$, $k \in \{1, \ldots, \prod_{i=1}^a d_i\}$, given by $(\operatorname{vec}\mathbf{T})_k = \mathbf{T}_{i_1,\ldots,i_a}$ with $i_j \in \{1, \ldots, d_j\}$ uniquely defined by $k = 1 + \sum_{j=1}^a (i_j - 1) \prod_{k=1}^{j-1} d_k$. Let $d_i = n$, $i = 1, \ldots, a$. The linear operator diag maps $\mathbf{T}$ to the vector $\operatorname{diag}\mathbf{T}$ in $\mathbb{R}^n$ with entry $(\operatorname{diag}\mathbf{T})_k$, $k \in \{1, \ldots, n\}$, given by $\mathbf{T}_{k,k,\ldots,k}$. If restricted to the set D of diagonal matrices, $D \ni \mathbf{D} \to \operatorname{diag}\mathbf{D}$ is bijective, and therefore the inverse mapping diag^{-1} is well-defined. Let $a = 2$ and $d_1 = d_2 = n$. The linear operator uni maps $\mathbf{T}$ to the vector $\operatorname{uni}\mathbf{T} \in \mathbb{R}^{n(n+1)/2}$ with entry $(\operatorname{vec}\mathbf{T})_k$, $k \in \{1, \ldots, n(n + 1)/2\}$, given by $(\operatorname{vec}\mathbf{T})_k = \mathbf{T}_{i,j}$ uniquely defined by $k = (j - 1)n - j(j - 1)/2 + i$. This operator maps a matrix $\mathbf{T}$ to a vector containing the entries of $\mathbf{T}$ read along its columns, but ignoring the elements above the main diagonal.

Following the notation in [1], the (n, n) identity matrix is denoted by $\mathbf{I}_n$, $\mathbf{C}_n$ denotes the (n^2, n^2) commutation matrix, i.e., the unique (n^2, n^2) matrix such that $\operatorname{vec}\mathbf{T}^{\mathrm{T}} = \mathbf{C}_n \operatorname{vec}\mathbf{T}$ for all (n, n) matrices $\mathbf{T}$. The matrix $\mathbf{D}_n$ denotes the $(n^2, n(n+1)/2)$ duplication matrix, i.e., the unique $(n^2, n(n+1)/2)$ matrix such that $\operatorname{vec}\mathbf{T} = \mathbf{D}_n \operatorname{uni}\mathbf{T}$ for all (n, n) symmetric matrices $\mathbf{T}$. From this definition, we have $\operatorname{uni}\mathbf{T} = \mathbf{D}_n^+ \operatorname{vec}\mathbf{T}$, where $\mathbf{A}^+$ indicates the Moore-Penrose inverse of the matrix $\mathbf{A}$. The duplication matrix and the commutation matrix are related through the identity $\mathbf{D}_n\mathbf{D}_n^+ = (1/2)(\mathbf{I}_{n^2} + \mathbf{C}_n)$. Finally, $\mathbf{1}_{n,m}$ denotes an (n, m) matrix of ones.

A *random scalar field* in the set X is a stochastic process, i.e., an indexed set $\mathcal{F}_X = \{\mathcal{F}_{\mathbf{x}}, \mathbf{x} \in X\}$ of random variables $\mathcal{F}_{\mathbf{x}}$ defined over the same probability space $(\Omega, \sigma_\Omega, P)$. A *random tensor fields* is defined analogously, with $\mathcal{F}_{\mathbf{x}}$ a vector-valued function.

Let $\mathcal{F}_X^1$ and $\mathcal{F}_X^2$ be two random scalar fields as above, and assume that $\mathcal{F}_{\mathbf{x}}^i$ is zero mean, which, for the purposes of this work, implies no loss of generality. If the expectation $E\{\mathcal{F}_{\mathbf{x}}^1\mathcal{F}_{\mathbf{y}}^2\}$ taken over the joint distribution of $\mathcal{F}_X^1$ and $\mathcal{F}_X^2$ is defined for all $\mathbf{x}$ and $\mathbf{y}$ in X, the function $R_{\mathcal{F}^1,\mathcal{F}^2}(\mathbf{x}, \mathbf{y}) = E\{\mathcal{F}_{\mathbf{x}}^1\mathcal{F}_{\mathbf{y}}^2\}$ defines the *cross-covariance function* of the random fields. If $\mathcal{F}_X^1 = \mathcal{F}_X^2 = \mathcal{F}_X$, the notation is simplified to $R_{\mathcal{F}} \triangleq R_{\mathcal{F},\mathcal{F}}$, and the function $R_{\mathcal{F}}$ is referred to as the *autocovariance function* of $\mathcal{F}_X$. The *conditional autocovariance function* of $\mathcal{F}_X^1$ given $\mathcal{F}_X^2$, $R_{\mathcal{F}^1|\mathcal{F}^2}$, is defined by taking the expectation of $\mathcal{F}_{\mathbf{x}}^1$ over the conditional distribution of $\mathcal{F}_{\mathbf{x}}^1$ and $\mathcal{F}_{\mathbf{y}}^1$ given $\mathcal{F}_X^2$. In the case of a random tensor field the definitions are analogous, with the product $\mathcal{F}_{\mathbf{x}}^1\mathcal{F}_{\mathbf{y}}^2$ replaced by a tensor product $\operatorname{vec}\mathcal{F}_{\mathbf{x}}^1 \otimes (\operatorname{vec}\mathcal{F}_{\mathbf{y}}^2)^{\mathrm{T}}$ and the expectation taken over each entry of the tensor. Henceforth, the term "random field" will be used in reference to both scalar and tensorial random fields.

Let X be a vector space. Zero-mean random fields $\mathcal{F}_X^1$ and $\mathcal{F}_X^2$ on X are *wide-sense stationary* if their cross-covariance function $R_{\mathcal{F}_X^1,\mathcal{F}_X^2}(\mathbf{x}, y)$ satisfies $R_{\mathcal{F}_X^1,\mathcal{F}_X^2}(\mathbf{x}, y) = R_{\mathcal{F}_X}(\mathbf{s})$ with $\mathbf{s} = \mathbf{x} - y$ for all $\mathbf{x}$ and $\mathbf{y}$ in X. Henceforth we will assume that $X = \mathbb{R}^n$, and therefore the subscript X in $\mathcal{F}_X$ can then be safely omitted by defining $\mathcal{F} \triangleq \mathcal{F}_{\mathbb{R}^n}$. A stationary random field on $\mathbb{R}^n$ is *isotropic* if its autocovariance function $R_{\mathcal{F}}(\mathbf{s})$ satisfies $R_{\mathcal{F}}(\boldsymbol{s}) = \sigma^2\rho(||\mathbf{s}||)$, where ρ is a correlation function [2] and $||\cdot||$ is the standard Euclidean norm in $\mathbb{R}^n$.

II. DERIVATIVES OF ISOTROPIC RANDOM FIELDS

Great simplification is achieved in the derivations that follow if $\rho(||\mathbf{s}||)$ can be rewritten as $\rho(||\mathbf{s}||) = r(||\mathbf{s}||^2)/\sigma^2 \Leftrightarrow \rho(\sqrt{||\mathbf{s}||}) = r(||\mathbf{s}||)/\sigma^2$. For $||\mathbf{s}|| > 0$ the smoothness of $r(||\mathbf{s}||)$ is contingent upon that of $\rho(||\mathbf{s}||)$. However, for $||\mathbf{s}|| = 0$ the non-differentiability of $\sqrt{||\mathbf{s}||}$ could be a problem. This is not the case, as shown in appendix. The symbols $\rho_0^{(i)}$ and $r_0^{(i)}$ denotes the i-th derivative of $\rho(||\mathbf{s}||)$ and $r(||\mathbf{s}||)$ with respect to $s = ||\mathbf{s}||$ at $s = 0$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function. The symbol $\frac{\partial^{a+b} f}{\partial \mathbf{x}^{a\mathrm{T}} \partial \mathbf{x}^b}$ is used to describe the matrix of dimensions (n^a, n^b) of partial derivatives of f, i.e,

$$\left(\frac{\partial^{a+b} f}{\partial \mathbf{x}^{a\mathrm{T}} \partial \mathbf{x}^b}\right)_{A,B} \triangleq \frac{\partial^{a+b} f}{\partial x_{a_1} \dots \partial x_{a_a} \partial x_{b_1} \dots \partial x_{b_b}},$$

where $a_i \in \{1, 2, \dots, n\}$ and $b_j \in \{1, 2, \dots, n\}$ are uniquely defined by $A = 1 + \sum_{i=1}^n (a_i - 1)n^{i-1}$ and $B = 1 + \sum_{j=1}^n (b_j - 1)n^{j-1}$. Second differentiability of the autocovariance function $R_{\mathcal{F}}(\mathbf{x}, \mathbf{y})$ of a random field at the pair $(\mathbf{x}, x)$ implies *mean square differentiability* of the random field itself, as demonstrated in theorem 2.4 of [2]. If a mean-square differentiable random field is stationary, its derivatives will also be stationary. For $\mathcal{F}^1$ and $\mathcal{F}^2$ jointly stationary with cross-covariance $R_{\mathcal{F}^1,\mathcal{F}^2}(\mathbf{s})$, we define $R^0_{\mathcal{F}^1,\mathcal{F}^2} \triangleq R_{\mathcal{F}^1,\mathcal{F}^2}(\mathbf{0})$.

The theorem that follows is central to this work:

Theorem 1. *Let $\mathcal{F}$ be an isotropic random field on $\mathbb{R}^n$ with autocovariance function $R_{\mathcal{F}}(\boldsymbol{s}) = \sigma^2\rho(||\mathbf{s}||)$ where $\rho(s)$ is four-differentiable. Let $\partial\mathcal{F}$ and $\partial^2\mathcal{F}$ be the tensor fields defined as*

$$\partial\mathcal{F} \triangleq \frac{\partial\mathcal{F}}{\partial \mathbf{x}^{\mathrm{T}}} \quad \textit{and} \quad \partial^2\mathcal{F} \triangleq \frac{\partial^2\mathcal{F}}{\partial \mathbf{x}^{\mathrm{T}} \partial \mathbf{x}}.$$

Then

$$R^0_{\partial\mathcal{F}} = -\sigma^2\rho_0^{(2)}\mathbf{I}_n, \tag{1a}$$

$$R^0_{\partial\mathcal{F},\partial^2\mathcal{F}} = \mathbf{O}_{n,n^2}, \tag{1b}$$

$$R^0_{\partial^2\mathcal{F}} = \sigma^2\rho_0^{(4)}(\mathbf{I}_{n^2} + \mathbf{C}_n + \operatorname{vec}\mathbf{I}_n \operatorname{vec}^{\mathrm{T}}\mathbf{I}_n). \tag{1c}$$

Proof. Using lemma 4 in appendix we write $\rho(\|\mathbf{s}\|) = r(\|\mathbf{s}\|^2)/\sigma^2$. From the four-differentiability of r and theorem 2.4 in [2] we have

$$R_{\partial\mathcal{F}}(\boldsymbol{s}) = -\frac{\partial^2 r(\|\mathbf{s}\|^2)}{\partial\boldsymbol{s}^{\mathrm{T}}\partial\boldsymbol{s}}, \tag{2a}$$

$$R_{\partial\mathcal{F},\partial^2\mathcal{F}}(\boldsymbol{s}) = -\frac{\partial^3 r(\|\mathbf{s}\|^2)}{\partial\boldsymbol{s}^{\mathrm{T}}\partial\boldsymbol{s}^2}, \tag{2b}$$

$$R_{\partial^2\mathcal{F}}(\boldsymbol{s}) = \frac{\partial^4 r(\|\mathbf{s}\|^2)}{\partial\boldsymbol{s}^{2\mathrm{T}}\partial\boldsymbol{s}^2}. \tag{2c}$$

The chain rule can be used to expand (2a)–(2c), and substituting the identities $\frac{\partial\mathbf{x}}{\partial\mathbf{x}} = \frac{\partial\mathbf{x}^{\mathrm{T}}}{\partial\mathbf{x}^{\mathrm{T}}} = \mathbf{I}_n$, $\frac{\partial(\mathbf{x}^{\mathrm{T}}\otimes\mathbf{I}_n)}{\partial\mathbf{x}^{\mathrm{T}}} = \mathbf{I}_n\otimes\mathbf{I}_n$ and $\frac{\partial(\mathbf{I}_n\otimes\mathbf{x}^{\mathrm{T}})}{\partial\mathbf{x}^{\mathrm{T}}} = \mathbf{C}_n$ in the result produces

$$\begin{aligned}
R_{\partial\mathcal{F}}(\boldsymbol{s}) &= -4r^{(2)}(\|\mathbf{s}\|^2)\boldsymbol{s}\otimes\boldsymbol{s}^{\mathrm{T}} - 2r^{(1)}(\|\mathbf{s}\|^2)\mathbf{I}_n,\\
R_{\partial\mathcal{F},\partial^2\mathcal{F}}(\boldsymbol{s}) &= -4\{2r^{(3)}(\|\mathbf{s}\|^2)\boldsymbol{s}\otimes\boldsymbol{s}^{\mathrm{T}}\otimes\boldsymbol{s}^{\mathrm{T}}+\\
&\quad r^{(2)}(\|\mathbf{s}\|^2)(\boldsymbol{s}^{\mathrm{T}}\otimes\mathbf{I}_n+\\
&\quad \mathbf{I}_n\otimes\boldsymbol{s}^{\mathrm{T}} + \boldsymbol{s}\otimes\operatorname{vec}^{\mathrm{T}}\mathbf{I}_n)\},\\
R_{\partial^2\mathcal{F}}(\boldsymbol{s}) &= 4\{8r^{(4)}(\|\mathbf{s}\|^2)\boldsymbol{s}\otimes\boldsymbol{s}\otimes\boldsymbol{s}^{\mathrm{T}}\otimes\boldsymbol{s}^{\mathrm{T}}+\\
&\quad 8r^{(3)}(\|\mathbf{s}\|^2)(\operatorname{vec}\mathbf{I}_n\otimes\boldsymbol{s}^{\mathrm{T}}\otimes\boldsymbol{s}^{\mathrm{T}}+\\
&\quad 2\boldsymbol{s}\otimes(\mathbf{I}_n\otimes\boldsymbol{s}^{\mathrm{T}} + \boldsymbol{s}^{\mathrm{T}}\otimes\mathbf{I}_n) + \boldsymbol{s}\otimes\boldsymbol{s}\otimes\operatorname{vec}\mathbf{I}_n^{\mathrm{T}})+\\
&\quad r^{(2)}(\|\mathbf{s}\|^2)(\mathbf{I}_n\otimes\mathbf{I}_n + \mathbf{C}_n + \operatorname{vec}\mathbf{I}_n\otimes\operatorname{vec}^{\mathrm{T}}\mathbf{I}_n)\}.
\end{aligned}$$

Making $\boldsymbol{s} = \mathbf{0}$ completes the proof. □

III. CURVATURES OF GAUSSIAN RANDOM FIELDS

Let $f(\mathbf{x})$ be a second-differentiable scalar function on $\mathbb{R}^n$. For each $\mathbf{x}\in\mathbb{R}^n$ for which $\partial f/\partial\mathbf{x} \neq \mathbf{0}$ we define the set $F_{\mathbf{x}}$ as $F_{\mathbf{x}} = \{\mathbf{y}\in\mathbb{R}^n \text{ such that } f(\mathbf{y}) = f(\mathbf{x}) \text{ and } \partial f/\partial\mathbf{y} \neq \mathbf{0}\}$. If $F_{\mathbf{x}} \neq \emptyset$, $F_{\mathbf{x}}$ is a $(n-1)$-hypersurface in $\mathbb{R}^n$ [3]. Let $\partial f \triangleq \partial f/\partial\mathbf{x}^{\mathrm{T}}$ and $\partial^2 f \triangleq \partial^2 f/\partial\mathbf{x}^{\mathrm{T}}\partial\mathbf{x}$. The *principal*

curvatures of the hypersurface $F_{\mathbf{x}}$ at $\mathbf{x}$ are given by the set of eigenvalues κ obtained by solving the eigenproblem

$$-\left(\mathbb{I}_n - \frac{\partial f \otimes (\partial f)^{\mathrm{T}}}{\|\partial f\|^2}\right)\frac{\partial^2 f}{\|\partial f\|}\bigg|_{\mathbf{x}} \mathbf{v} = \kappa \mathbf{v}, \tag{4}$$

with $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v}\| = 1$ and $\mathbf{v}^{\mathrm{T}}\partial f = 0$ [4, pg. 138]. Let $\{\mathbf{n}_i, i = 1, \ldots, n-1\}$ be an orthornormal basis for the null-space of ∂f, i.e., $\mathbf{n}_i^{\mathrm{T}}\partial f = 0$ and $\mathbf{n}_i^{\mathrm{T}}\mathbf{n}_j = \delta_{ij}$, and let $\mathbf{N}$ be the matrix $\mathbf{N} = [\mathbf{n}_1 \ldots \mathbf{n}_{n-1}]$. The eigenproblem in (4) can be rewritten as

$$-\frac{\mathbf{N}^{\mathrm{T}}(\partial^2 f)\mathbf{N}}{\|\partial f\|}\bigg|_{\mathbf{x}} \mathbf{u} = \kappa \mathbf{u}, \tag{5}$$

with $\mathbf{u} \in \mathbb{R}^{n-1}$, $\|\mathbf{u}\| = 1$. Equation (5) is still valid if the function f is the realization $\mathcal{F}(\omega)$, $\omega \in \Omega$, of a mean-square second-differentiable scalar random field $\mathcal{F}$ on $\mathbb{R}^n$. Therefore the random tensor field $\mathcal{K}$ of curvatures of isotropic Gaussian random fields is implicitly defined at $\mathbf{x}$ such that $\partial\mathcal{F}_{\mathbf{x}}(\omega) \neq \mathbf{0}$ by the solutions of the equation

$$-\frac{\mathcal{N}^{\mathrm{T}}(\partial^2\mathcal{F})\mathcal{N}}{\|\partial\mathcal{F}\|}\mathcal{U} = \mathcal{K}\mathcal{U}, \tag{6}$$

where $\mathcal{N}$ is a random tensor field satisfying $\mathcal{N}_{\mathbf{x}}^{\mathrm{T}}\partial\mathcal{F}_{\mathbf{x}} = 0$, and $\mathcal{N}_{\mathbf{x}}^{\mathrm{T}}\mathcal{N}_{\mathbf{x}} = \mathbf{I}_{n-1}$, and $\mathcal{U}$ is the tensor field such that $\mathcal{U}_{\mathbf{x}}(\omega)$ are the eigenvectors associated to the eigenvalues $\mathcal{K}_{\mathbf{x}}(\omega)$.

Henceforth we assume that $\mathcal{F}$ is an isotropic, second-differentiable *Gaussian random field*, which is defined simply as an isotropic random field for which the joint distribution of any finite set of random variables $\{\mathcal{F}_A\}$, $A \in \mathbb{R}^n$, is Gaussian. This assumption implies that the zero-mean random tensors $\partial\mathcal{F}$ and $\partial^2\mathcal{F}$ are also Gaussian, and therefore $\partial\mathcal{F}_{\mathbf{x}}$ and $\partial^2\mathcal{F}_{\mathbf{x}}$ are fully characterized by their covariance matrices, given by (1a) and (1c) in theorem 1. However, because $\partial^2\mathcal{F}$ is symmetric, its probability density must be handled with care, since $R^0_{\partial^2\mathcal{F}}$ is not invertible.

The following lemma is a trivial corollary of the theorems in [1, sec. 7].

Lemma 1. *Let* $\mathbf{A}_n$ *be a* (n, n) *invertible matrix. Then*

$$(\mathbf{A}_n \otimes \mathbf{A}_n)R^0_{\partial^2\mathcal{F}}(\mathbf{A}_n^{-1} \otimes \mathbf{A}_n^{-1}) = R^0_{\partial^2\mathcal{F}}. \tag{7}$$

Let $\partial^2\mathcal{F}$ be as in theorem 1, and let $\mathcal{R}$ be a (n, m), $n \geq m$, orthonormal tensor field independent of $\partial^2\mathcal{F}$, i.e., a random tensor field such that for all $\mathbf{x} \in \mathbb{R}^n$ any realization $\mathcal{R}(\omega)$ of $\mathcal{R}$ satisfies $\mathcal{R}_{\mathbf{x}}^{\mathrm{T}}(\omega)\mathcal{R}_{\mathbf{x}}(\omega) = \mathbf{I}_m$ and $\mathcal{R}_{\mathbf{x}}$ is independent of $\partial^2\mathcal{F}_{\mathbf{x}}$. Define $\partial^2\mathcal{F}' \triangleq \mathcal{R}^{\mathrm{T}}\partial^2\mathcal{F}\mathcal{R} = \{\mathcal{R}_{\mathbf{x}}{}^{\mathrm{T}}\partial^2\mathcal{F}_{\mathbf{x}}\mathcal{R}_{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^n\}$. We prove the following lemma:

Lemma 2. *$\partial^2\mathcal{F}'$ is a Gaussian random tensor field with autocovariance function $R_{\partial^2\mathcal{F}'}(\mathbf{s})$ such that*

$$R^0_{\partial^2\mathcal{F}'} = \sigma^2\rho_0^{(4)}(\mathbf{I}_{m^2} + \mathbf{C}_m + \operatorname{vec}\mathbf{I}_m \operatorname{vec}^{\mathrm{T}}\mathbf{I}_m). \tag{8}$$

Proof. Let $P_{\partial^2\mathcal{F}'_{\mathbf{x}}|\mathcal{R}_{\mathbf{x}}}$ be the conditional probability of the random tensor $\partial^2\mathcal{F}'_{\mathbf{x}}$ given $\mathcal{R}_{\mathbf{x}}$. Since $\partial^2\mathcal{F}_{\mathbf{x}}$ is independent of $\mathcal{R}_{\mathbf{x}}$, $\partial^2\mathcal{F}'_{\mathbf{x}}$ given $\mathcal{R}_{\mathbf{x}}$ is a linear function of $\partial^2\mathcal{F}_{\mathbf{x}}$, and therefore it is zero-mean Gaussian. Using the identity $\operatorname{vec}(\mathbf{A}BC) = (\mathbf{C}^{\mathrm{T}} \otimes \mathbf{A})\operatorname{vec}\mathbf{B}$ and properties of commutator matrices [1] we can write $\operatorname{vec}\partial^2\mathcal{F}'_{\mathbf{x}} = (\mathcal{R}_{\mathbf{x}}{}^{\mathrm{T}} \otimes \mathcal{R}_{\mathbf{x}}{}^{\mathrm{T}})\operatorname{vec}\partial^2\mathcal{F}_{\mathbf{x}}$, and therefore

$$\begin{aligned} R^0_{\partial^2\mathcal{F}'_{\mathbf{x}}|\mathcal{R}_{\mathbf{x}}} &= (\mathcal{R}_{\mathbf{x}}{}^{\mathrm{T}} \otimes \mathcal{R}_{\mathbf{x}}{}^{\mathrm{T}})R^0_{\partial^2\mathcal{F}}(\mathcal{R}_{\mathbf{x}} \otimes \mathcal{R}_{\mathbf{x}}) \qquad (9)\\ &= \sigma^2\rho_0^{(4)}(\mathbf{I}_{m^2} + \mathbf{C}_m + \operatorname{vec}\mathbf{I}_m \operatorname{vec}^{\mathrm{T}}\mathbf{I}_m), \qquad (10)\end{aligned}$$

using lemma 1. Since $\partial^2\mathcal{F}'_{\mathbf{x}}$ given $\mathcal{R}_{\mathbf{x}}$ is zero-mean Gaussian and $R^0_{\partial^2\mathcal{F}'|\mathcal{R}_{\mathbf{x}}}$ does not depend on $\mathcal{R}_{\mathbf{x}}$ for fixed m and n, we have $P_{\partial^2\mathcal{F}'_{\mathbf{x}}|\mathcal{R}_{\mathbf{x}}} = P_{\partial^2\mathcal{F}'_{\mathbf{x}}}$. Therefore $P_{\partial^2\mathcal{F}'_{\mathbf{x}}}$ is zero-mean Gaussian with autocovariance function satisfying (8). ◻

Lemma 2 justifies the notation $\partial^2\mathcal{F}^m \triangleq \mathcal{R}^{\mathrm{T}}\partial^2\mathcal{F}\mathcal{R}$, for $\mathcal{R}$ (n, m). Let $\delta^2\mathcal{F}^n$ be the $(n(n+1)/2, 1)$ vector defined as $\delta^2\mathcal{F}^n \triangleq \operatorname{uni}\partial^2\mathcal{F}^n = \mathbf{D}_n^+ \operatorname{vec}\partial^2\mathcal{F}^n$. Its covariance matrix $\mathbf{\Sigma}_n$ is invertible and given by

$$\mathbf{\Sigma}_n = \mathbf{D}_n^+ R^0_{\partial^2\mathcal{F}^n}\mathbf{D}_n^{+,\mathrm{T}}. \tag{11}$$

Therefore the probability density $p_{\partial^2\mathcal{F}^n_{\mathbf{x}}}$ of $\partial^2\mathcal{F}^n_{\mathbf{x}}$ is standard:

$$p_{\delta^2\mathcal{F}^n_{\mathbf{x}}}(\mathbf{h}) = \frac{1}{\sqrt{|2\pi\mathbf{\Sigma}_n|}}\exp\Big(-\frac{\mathbf{h}^{\mathrm{T}}\mathbf{\Sigma}_n^{-1}\mathbf{h}}{2}\Big). \tag{12}$$

We now define the random fields $(\mathcal{R}^n, \mathcal{L}^n) \triangleq \{(\mathcal{R}^n_{\mathbf{x}}, \mathcal{L}^n_{\mathbf{x}}) \in SO(n) \times \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^n | \ \mathcal{R}^{n,\mathrm{T}}_{\mathbf{x}}\mathcal{R}^n_{\mathbf{x}} = \mathbf{I}_n$ and $\mathcal{R}^{n,\mathrm{T}}_{\mathbf{x}} \operatorname{diag}^{-1}\mathcal{L}^n_{\mathbf{x}}\mathcal{R}^n_{\mathbf{x}} = \partial^2\mathcal{F}^n_{\mathbf{x}}\}$, where $SO(n)$ is the special orthogonal group of (n, n) matrices. Let $\operatorname{eig}_n^{-1}$ be the mapping given by

$$\begin{aligned} \operatorname{eig}_n^{-1} : SO(n) \times \mathbb{R}^n &\to \mathbb{S}(n) \\ (\mathbf{R}, \mathbf{\Lambda}) &\mapsto \mathbf{S} = \operatorname{eig}_n^{-1}(\mathbf{R}, \mathbf{\Lambda}) \end{aligned}, \tag{13}$$

where $\mathbb{S}(n)$ is the set of (n, n) symmetric matrices. This mapping is differentiable and onto, and therefore the joint probability density $p_{\mathcal{R}_{\mathbf{x}},\mathcal{L}_{\mathbf{x}}}$ of $\mathcal{R}_{\mathbf{x}}$ and $\mathcal{L}_{\mathbf{x}}$ is given by

$$p_{\mathcal{R}^n_{\mathbf{x}},\mathcal{L}^n_{\mathbf{x}}}(\mathbf{R}, \lambda) = p_{\partial^2\mathcal{F}^n_{\mathbf{x}}}(\operatorname{uni}(\mathbf{R}^{\mathrm{T}} \operatorname{diag}^{-1}\lambda\mathbf{R}))|J(\mathbf{R}, \lambda)|, \tag{14}$$

$|J(\mathbf{R}, \lambda)|$ is the absolute value of the Jacobian determinant of eig_n^{-1}.

Theorem 3.3.1 in [5] provides a "closed-form" expression of the probability density of the eigenvalues of random matrices in the *Gaussian orthogonal ensemble* (GOE_n). This is the ensemble of (n, n) real symmetric matrices $\mathcal{M}$ with probability density invariant with respect to similarity transformations $\mathcal{M} \to \mathbf{R}^{\mathrm{T}} \mathcal{M} R$ for any given (n, n) orthonormal $\mathbf{R}$ *and* such that the probability distribution of distinct entries are independent from each other. Even though each realization of $\partial^2 \mathcal{F}^n$ is real, symmetric, and, by applying lemma 2, invariant to the same similarity transformations, $\partial^2 \mathcal{F}^n$ is different from GOE_n, because the distinct entries of $\partial \mathcal{F}_{\mathbf{x}}^n$ are not independent. However, the assumption of independency used in [5] is important only to derive an expression for the joint probability density of the entries of random matrices in GOE_n, and we already have that for the matrices in $\delta^2 \mathcal{F}^n$. Once such an expression is available the result in [5] derives from the observation that, in an expression analogous to (14), the term $\mathbf{R}$ appeared only on $|J(\mathbf{R}, \lambda)|$, and therefore the probability density of the eigenvalues of matrices in GOE_n is obtained through the integration of $|J(\mathbf{R}, \lambda)|$ over $SO(n)$. The next lemma shows that this is also the case for the probability density $p_{\mathcal{L}_{\mathbf{x}}^n}$ of the eigenvalues of $\partial^2 \mathcal{F}_{\mathbf{x}}^n$:

Lemma 3. *Let $\lambda \in \mathbb{R}^n$, $\mathbf{R} \in SO(n)$, and $\mathbf{\Sigma}_n$ as in (11). Then*

$$p_{\delta^2 \mathcal{F}_{\mathbf{x}}^n}(\mathrm{uni}(\mathbf{R}^{\mathrm{T}} \operatorname{diag}^{-1} \lambda \mathbf{R})) = \frac{1}{\sqrt{|2\pi \mathbf{\Sigma}_n|}} \exp\left(- \frac{\lambda^{\mathrm{T}} \tilde{\mathbf{\Sigma}}_n^{-1} \lambda}{2} \right), \tag{15}$$

where $\tilde{\mathbf{\Sigma}}_n = \sigma^2 \rho_0^{(4)} (2\mathbf{I}_n + \mathbf{1}_{n,1} \mathbf{1}_{n,1}^{\mathrm{T}})$.

Proof. The following identity can be easily verified:

$$(R^0_{\partial^2 \mathcal{F}^n})^+ = \frac{1}{4\sigma^2 \rho_0^{(4)}} \left(\mathbf{I}_{n^2} + \mathbf{C}_n - \frac{2}{2+n} \operatorname{vec} \mathbf{I}_n \operatorname{vec}^{\mathrm{T}} \mathbf{I}_n \right). \tag{16}$$

Let $\mathbf{\Lambda} = \operatorname{diag}^{-1} \lambda$ and $\mathbf{u}_{\mathbf{R}} \triangleq \mathrm{uni}(\mathbf{R}^{\mathrm{T}} \mathbf{\Lambda} \mathbf{R})$. Therefore,

$$\begin{aligned}
\mathbf{u}_R{}^{\mathrm{T}} \mathbf{\Sigma}_n^{-1} \mathbf{u}_R &= [\mathrm{uni}(\mathbf{R}^{\mathrm{T}} \mathbf{\Lambda} \mathbf{R})]^{\mathrm{T}} \mathbf{\Sigma}_n^{-1} [\mathrm{uni}(\mathbf{R}^{\mathrm{T}} \mathbf{\Lambda} \mathbf{R})] \\
&= [\mathbf{D}_n^+ \operatorname{vec}(\mathbf{R}^{\mathrm{T}} \mathbf{\Lambda} \mathbf{R})]^{\mathrm{T}} \mathbf{\Sigma}_n^{-1} [\mathbf{D}_n^+ \operatorname{vec}(\mathbf{R}^{\mathrm{T}} \mathbf{\Lambda} \mathbf{R})] \\
&= (\operatorname{vec} \mathbf{\Lambda})^{\mathrm{T}} (\mathbf{R} \otimes \mathbf{R}) \mathbf{D}_n^{+,\mathrm{T}} \mathbf{\Sigma}_n^{-1} \mathbf{D}_n^+ (\mathbf{R}^{\mathrm{T}} \otimes \mathbf{R}^{\mathrm{T}}) \operatorname{vec} \mathbf{\Lambda},
\end{aligned}$$

and, using $[(\mathbf{R} \otimes \mathbf{R}) \mathbf{D}_n]^+ = \mathbf{D}_n^+ (\mathbf{R}^{\mathrm{T}} \otimes \mathbf{R}^{\mathrm{T}})$,

$$\mathbf{u}_R{}^{\mathrm{T}} \mathbf{\Sigma}_n^{-1} \mathbf{u}_R = (\operatorname{vec} \mathbf{\Lambda})^{\mathrm{T}} [(\mathbf{R} \otimes \mathbf{R}) \mathbf{D}_n \mathbf{\Sigma}_n \mathbf{D}_n^{\mathrm{T}} (\mathbf{R}^{\mathrm{T}} \otimes \mathbf{R}^{\mathrm{T}})]^+ \operatorname{vec} \mathbf{\Lambda},$$

which, using $\mathbf{D}_n\boldsymbol{\Sigma}_n\mathbf{D}_n^{\mathrm{T}} = R^0_{\partial^2\mathcal{F}^n}$, yields

$$\begin{aligned}
\mathbf{u}_R{}^{\mathrm{T}}\boldsymbol{\Sigma}_n^{-1}\mathbf{u}_R &= (\operatorname{vec}\boldsymbol{\Lambda})^{\mathrm{T}}[(\mathbf{R}\otimes\mathbf{R})R^0_{\partial^2\mathcal{F}^n}(\mathbf{R}^{\mathrm{T}}\otimes\mathbf{R}^{\mathrm{T}})]^{+}\operatorname{vec}\boldsymbol{\Lambda}\\
&= (\operatorname{vec}\boldsymbol{\Lambda})^{\mathrm{T}}(R^0_{\partial^2\mathcal{F}^n})^{+}\operatorname{vec}\boldsymbol{\Lambda},\\
&= \boldsymbol{\lambda}^{\mathrm{T}}\tilde{\boldsymbol{\Sigma}}_n^{-1}\boldsymbol{\lambda}
\end{aligned}\tag{17}$$

using lemma 1 and (16). □

The integration of $|J(\mathbf{R},\boldsymbol{\lambda})|$ over $SO(n)$, carried out in [5], gives

$$\int_{SO(n)}|J(\mathbf{R},\boldsymbol{\lambda})|\,d\mathbf{R} = \left(\frac{\pi^{(n+1)/4}}{2}\right)^n\frac{\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}|\lambda_j-\lambda_i|}{\prod_{i=1}^{n}\Gamma(1+i/2)},\tag{18}$$

and it can be shown that the determinant of $\boldsymbol{\Sigma}_n$ is given by

$$|\boldsymbol{\Sigma}_{\boldsymbol{n}}| = 2^{n-1}(2+n)(\sigma^2\rho_0^{(4)})^{n(n+1)/2}.\tag{19}$$

Together with lemma 3, these results demonstrate the following theorem:

Theorem 2. *The probability distribution $p_{\mathcal{L}^n_{\mathbf{x}}}$ of the eigenvalues of $\partial^2\mathcal{F}^n_{\mathbf{x}}$ is*

$$\begin{aligned}
p_{\mathcal{L}^n_{\mathbf{x}}}(\boldsymbol{\lambda}) = {}&\frac{2^{(2-7n-n^2)/4}}{\sqrt{2+n}(\sigma^2\rho_0^{(4)})^{n(n+1)/4}\prod_{i=1}^{n}\Gamma(1+i/2)}\times\\
&\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}|\lambda_j-\lambda_i|\exp\left(-\frac{\boldsymbol{\lambda}^{\mathrm{T}}\tilde{\boldsymbol{\Sigma}}_n^{-1}\boldsymbol{\lambda}}{2}\right).
\end{aligned}\tag{20}$$

Since $\mathcal{N}_{\mathbf{x}}$ in (6) is a function of $\partial\mathcal{F}^n_{\mathbf{x}}$, theorem 1(1b) implies that $\partial^2\mathcal{F}^n_{\mathbf{x}}$ is independent of $\mathcal{N}_{\mathbf{x}}$ for all $\mathbf{x}$. Therefore $\partial\mathcal{F}^{n-1}_{\mathbf{x}} = \mathcal{N}_{\mathbf{x}}\partial\mathcal{F}^n_{\mathbf{x}}\mathcal{N}_x$ according to lemma 2. Theorem 2 can then be applied to obtain an expression for the probability density of the eigenvalues of the numerator of (6), $p_{\mathcal{L}^{n-1}_{\mathbf{x}}}$. Using theorem 1(1a), we can show that the denominator of (6), $\|\partial\mathcal{F}^n_{\mathbf{x}}\|$, is distributed according to $\sigma(-\rho_0^{(0)})^{1/2}\mathcal{X}(n)$, where $\mathcal{X}(n)$ follows a *χ-distribution* with n degrees of freedom, and therefore its probability density $p_{\|\partial\mathcal{F}^n_{\mathbf{x}}\|}$ is given by

$$p_{\|\partial\mathcal{F}^n_{\mathbf{x}}\|}(u) = \frac{2u^{n-1}\exp[u^2/(2\sigma^2\rho_0^{(2)})]}{(-2\sigma^2\rho_0^{(2)})^{n/2}\Gamma(n/2)},\tag{21}$$

We can now prove our main result:

Theorem 3. *Let $\mathcal{K}$ be as in (6). Then*

$$\begin{aligned}
p_{\mathcal{K}_{\mathbf{x}}}(\boldsymbol{\kappa}) = {}&\frac{2^{(n^2-7n+8)/4}\Gamma[n(n+1)/4]}{\sqrt{1+n}\,\Gamma(n/2)\prod_{i=1}^{n-1}\Gamma(1+i/2)}\times\\
&\frac{\alpha^{n(n-1)/4}\prod_{i=1}^{n-2}\prod_{j=i+1}^{n-1}|\kappa_j-\kappa_i|}{\{\alpha[\sum_{i=1}^{n-1}\kappa_i^2-\frac{1}{n+1}(\sum_{i=1}^{n-1}\kappa_i)^2]+1\}^{\frac{n^2+n}{4}}},
\end{aligned}\tag{22}$$

where $\alpha = -\rho_0^{(2)}/(2\rho_0^{(4)})$.

Proof. Since $\partial\mathcal{F}_{\mathbf{x}}^{n}$ and $\partial^2\mathcal{F}_{\mathbf{x}}^{n-1}$ are independent, so will be $\|\partial\mathcal{F}_{\mathbf{x}}^{n}\|$ and $\mathcal{L}_{\mathbf{x}}^{n-1}$. Using (6), we have.

$$p_{\mathcal{K}_{\mathbf{x}}}(\boldsymbol{\kappa}) = \int_0^\infty u^{n-1} p_{\mathcal{L}_{\mathbf{x}}^{n-1}}(\boldsymbol{\kappa} u) p_{\|\partial\mathcal{F}_{\mathbf{x}}^{n}\|}(u)\, du. \tag{23}$$

Substituting (20) and (21) in (23), we obtain (22). □

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APPENDIX: DIFFERENTIABILITY OF THE AUTOCORRELATION FUNCTION

Correlation functions are characterized by the Wiener-Khintchine theorem, a simplified version off which, shown below, is quoted verbatim from [2]:

Theorem 4. *A real function $\rho(\|\mathbf{s}\|)$ on $\mathbb{R}^n$ is a correlation function if and only if it can be represented in the form*

$$\rho(\|\mathbf{s}\|) = 2^{(d-2)/2}\Gamma(d/2) \int_0^\infty \frac{J_{(d-2)/2}(ks)}{(ks)^{(d-2)/2}}\, d\Phi(k), \tag{A.1}$$

where the function $\Phi(k)$ on $\mathbb{R}$ has the properties of a distribution function and J_ν is a Bessel functions of the first kind and order ν.

Lemma 4. *Let the i-th moment of the distribution $\Phi(k)$ in theorem 4 be defined. Then, the i-th derivative of $r(\|\mathbf{s}\|)$, $r^{(i)}(\|\mathbf{s}\|)$, exists and is given by*

$$r^{(i)}(\|\mathbf{s}\|) = 2^{(d-2)/2}\Gamma(d/2) \int_0^\infty k^i \frac{J_{(d-2)/2}(ks)}{(ks)^{(d-2)/2}}\, d\Phi(k). \tag{A.2}$$

Proof. Define the operator D_i acting on a function $f(u)$ as

$$D_i[f(u)] = \left(\frac{1}{u}\frac{d}{du}\right)_i [f(u)] \tag{A.3}$$

where the term in the right-hand side is recursively defined as

$$\left(\frac{1}{u}\frac{d}{du}\right)_1 [f(u)] = \frac{1}{u}\frac{d\,f(u)}{du} \tag{A.4}$$

$$\left(\frac{1}{u}\frac{d}{du}\right)_i [f(u)] = \left(\frac{1}{u}\frac{d}{du}\right)\left[\left(\frac{1}{u}\frac{d}{du}\right)_{i-1} [f(u)]\right]. \tag{A.5}$$

It can be shown by induction that

$$r^{(i)}(\|\mathbf{s}\|) = \left(\frac{1}{u}\frac{d}{du}\right)_i [\rho(u)]\bigg|_{u=\sqrt{\|\mathbf{s}\|}}. \tag{A.6}$$

The operator D_i and the integral in theorem 4 can be interchanged, since the functions and the measure $d\Phi(k)$ involved satisfy the conditions of Lebesgue's dominated convergence theorem. The identity

$$D_i\left[\frac{J_\nu(u)}{u^\nu}\right] = (-1)^i \frac{J_{\nu+i}(u)}{u^{\nu+i}}, \tag{A.7}$$

found in [6], completes the proof. □

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